Job Security, Stability and Production Efficiency*
PRELIMINARY DRAFT

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Abstract

We study a 2-sided labor market with a set of heterogeneous firms and workers in an environment where jobs are secured by regulation. Without job security Kelso and Crawford have shown that stable outcomes and efficiency prevail when all workers are (weak) gross substitutes to each firm, in the sense that increases in other workers' salaries can never cause a firm to withdraw an offer from a worker whose salary has not risen. It turns out that by allowing job security such stability and efficiency prevail under much weaker assumptions.

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1 Introduction

Since the work of Kelso and Crawford (1982) the 2-sided many-to-many matching model has emerged as the prominent tool to analyze labor markets whenever firms and workers are heterogeneous. The assumption underlying many results in this literature is that firms' preferences over sets of workers exhibit "gross-substitutability". The Kelso and Crawford model assumes that workers and firms negotiate over a single parameter, the worker's salary, and the particular details of the job are given (such as working hours, shifts, vacation days, insurance). Hatfield and Milgrom (2005) generalize this matching-based model to a 'contracts model' where a multi-dimensional set of feasible contracts between firms and workers exists. In such a model the productivity of a set of workers is not just based on their identity as in Kelso and Crawford but also on the nature of the contract. Hatfield and Milgrom go on and show that in this more general model gross-substitutability (suitably defined) drives many similar results (such as existence of non-empty core and efficiency). Echenique (2012) shows that under this assumption both models are equivalent.[1]

The notion of stability, initially due to Gale and Shapley (1962), is the standard solution concept for matching models in general and for labor markets in particular. A stable outcome is an allocation of workers to firms (of which one firm is the outside option of unemployment) and a salary vector for the workers such that no combination of a single firm and a set of workers can improve their position while disregarding the others (there is no 'blocking coalition'). Underlying the logic of this solution concept is the notion of a free, unregulated, competitive market, where any coalition can withdraw from the market if the market does not provide them with a desired outcome. However, what is the natural solution concept when markets are regulated, in particular when workers enjoy regulation related to employment security? The theoretical literature on matching seems to be mute about such issues. In this work we revise the notion of stability so it accounts for a regulated labor market. In particular we would like to model a regulated market where firms cannot unilaterally fire employees (or where such costs of firing are prohibitively high). In such labor markets a firm which belongs to a 'blocking coalition' must be accompanied by all of its current employees. In other words, regulation implies fewer blocking coalitions and consequently the requirements underlying the implied notion of stability, which we refer to as JS-stability (where JS stands for Job Security) becomes easier to satisfy.

It is no surprise, therefore, that we can guarantee the existence of JS-stable outcomes in markets where no stable outcomes exist. A key assumption for most results on labor markets is that of gross-substitutability. In the Kelso and Crawford model that we adopt, such gross-substitutability is a necessary and sufficient condition for a variety of results (see, among others, Kelso and Crawford

[1] Recently, Hatfield and Kojima (2010) show that one can obtain powerful theoretical results under weaker substitutability conditions, dubbed "bilateral" and "unilateral" substitutability. Sönmez and Switzer (2013) provide results on specific labor markets for cadets within the US army where the standard substitutability assumption does not hold. These works demonstrate the limitation of the Echenique critique regarding Hatfield and Milgrom's 'contracts model'.

(1982), Gul and Stacchetti (1999) and Ausubel (2006)). Our treatment, on the other hand, goes substantially beyond the scope of gross-substitutability and allows for a broader class of preferences. In fact, existence and optimality of a JS-stable outcome is guaranteed for the class of 'Almost Fractionally Sub-additive' valuations (AFS), which we formally define in the sequel. Furthermore, it is shown that this class is a maximal class for which such existence and optimality of a JS-stable allocation hold.

The 'gross substitutes' assumption, and in fact any assumption on substitutability, obviously rules out the treatment of markets where complementarities exist among the workers. Such complementarity is vital for the analysis of many particular markets. Two leading examples are those of the matching between players and sports clubs — as clubs are focused on generating a team spirit and building synergy among their players — and of the matching between universities and academics.[2] The class of AFS production functions, which is central to our analysis, allows for some limited form of complementarity among workers (see Example 1) and so we can undoubtedly argue that our work goes beyond substitutability.[3] In addition, it is straightforward to verify that the class of AFS production functions is a superset of the class of gross substitutes. In fact, it has been shown by Lehmann et al. (2006) to be substantially larger than the class of gross substitutes in some natural measure theoretic terms.[4]

1.1 Our contribution

Our contribution is conceptual as well as technical.

- Our conceptual contribution is two-fold:
    1. We introduce a new solution concept for the many-to-many matching model — JS-stability. This solution concept, tailored to analyze labor markets with employment protection, is inspired by the prevalence of regulation in many countries (in the EU in particular) which puts significant restrictions on firms' ability to fire employees. The on-going public debate of such regulation has not been part of the matching literature so far and JS-stability provides a (preliminary) means of introducing it.
    2. We introduce a new class of production functions, dubbed 'almost fractionally sub-additive' functions.
- For these new concepts we prove the following theorems:

[2]This paper might not have been conceived if Lavi and Smorodinsky or Fu and Kleinberg were members of the same department.

[3]In the neoclassical matching literature, starting with Becker (1973), the assumption of complementarities (otherwise known as 'positive assortative matching') is cardinal for many of the results. Note that the notion of complementarities assumed in that literature is related to the synergy between a firm and a (single) worker. In contrast, we refer to complementarities among workers with respect to a firms' production function.

[4]Technically, for a natural measure over the set of all production functions the measure of all the functions exhibiting gross-substituters has measure zero where the measure of the class AFS has positive measure.

1. We provide analogs of the welfare theorems to markets with job security. On the one hand, if firms' production functions are almost fractionally sub-additive (AFS) then any efficient outcome is sustained as a JS-stable outcome. On the other, although there may be inefficient JS-stable outcomes, we provide a bound on the efficiency loss such an outcome entails. In fact, in cardinal terms, summing over all players' utilities (as expressed with a numeraire good), any such outcome is at least 50% efficient.
2. We show that the family of AFS production functions is maximal with respect to obtaining our welfare theorems.
3. We provide a natural decentralized mechanism which yields a JS-stable outcome in equilibrium.

1.2 Literature on job security

The lion's share of the theoretical literature on job security and employment protection legislation makes use of partial and general equilibrium in dynamic models. A common thread of all these models is that the work force is assumed *homogeneous* (e.g., Gavin, 1986, Lazear, 1990, Acemoglu and Shimer, 2000, Bertola, 2004), which is in sharp contrast with our heterogeneity assumption. Typically, a firm's productivity depends on the size of the workforce but not on the exact composition of workers it employs. Whereas our model is static and with complete information these models are dynamic and information stochastically unravels with time (e.g., workers' productivity and firms' technology). Our paper also departs from the aforementioned body of literature in the main question posed. Whereas our focus is on existence and efficiency of a given solution concept (JS-stability) the aforementioned body of literature primarily attempts to study the impact of regulation and the introduction of job security measures on the unemployment level. On the one hand job security implies that fewer employees are fired but on the other hand employers are more cautious in their hiring process and are reluctant to employ low productivity employees. The study of these two countervailing forces is an important common thread of previous work.

Interestingly, the findings of this literature, both theoretically and empirically, are inconclusive. The exact nature of the model as well as the underlying assumptions, on the one hand and the econometric model deployed on the other hand lead to conclusions which are often inconsistent with each other. The reader is referred to a survey by Bertola (1999) to learn more about this.

1.3 Other matching markets

The newly introduced notion of JS-stability is primarily motivated by regulatory intervention designed to increase job security in labor markets. However, it may also have relevance in the study of immigration and community formation. In this context, firms are replaced by countries and workers by immigrants. Once citizenship is granted to an immigrant it is (almost) impossible to revoke. On the other hand, citizens have typically lower barriers to immigrate from their own country to another one. Thus, JS-stability may correctly represent the feasible community structure in a model of immigration.[5] In fact, there may be additional many-to-many matching markets where divorce costs on both sides of the market are highly asymmetric and so the notion of JS-stability becomes an adequate tool for their analysis.

1.4 Paper structure

Section 2 provides the model we employ and in particular details the new solution concept and production functions we introduce. One particular aspect of our model is that each worker requires a minimal wage in order to work for any given firm and this wage may differ across firms for each worker and definitely differs across workers. This is our way to model asymmetry of the firms from workers' perspective. Section 3, however, deals with a limited model in which such asymmetry is gone and the minimal wage is zero across all workers and all firms. It furthermore shows that this reduction is technically meaningless and results which are true for the specific model are true for the more general model with asymmetry. Section 4 provides the main results, and Section 5 discusses future research.

2 Model

A labor market is composed of a set of N firms and M workers such that each firm hires as many workers as it wishes, but each worker is allowed to work only at one firm. Each firm pays its workers a salary and the utility of each worker depends on which firm he works for and the salary he receives. The firms' objective function is their profit and each firm's profit is the difference between the value of its production (in salary units) and the salaries it pays out. Note, in particular, there are no externalities among workers nor among firms.

The formal model we use is due to Kelso and Crawford (1982) — A labor market is a tuple (N, M, v, b) where N is a finite set of firms and M is a finite set of workers (in the sequel we abuse notation and use N and M to denote the cardinality of these sets as well). $v = \{v^n\}_{n \in N}$, where $v^n : 2^M \to \Re_+$ is firm n's monotonically increasing production function, as measured in the same units as salaries.[6] We calibrate $v^n(\emptyset) = 0$. $b = \{b^n_m\}_{m \in M, n \in N}$, where $b^n_m \geq 0$ is the minimal salary requested by worker m in order to work for firm n. The utility of worker m for working at firm n at salary s is denoted $u_m(n, s) = s - b^n_m$.[7] Hereinafter firm 0 will denote unemployed workers and

[5] We thank Yoram Weiss for pointing out this connection between JS-stability and community formation.

[6] v^n is monotonically increasing if $C \subset D \implies v^n(C) \leq v^n(D)$.

[7] The model and results in Kelso and Crawford (1982) make use of an abstract utility function for workers which takes into account the firm they work for as well as their salary. This general form is not necessarily given in salary terms. In contrast, we restrict attention to additive-separable utility functions for two reasons. On the one hand, when workers', as well as firms', utility functions are all in salary terms there is a natural cardinal notion of efficiency,

we calibrate $b^0_m = 0$ for all m. As productivity is measured in salary units, the profit of firm n from employing a set of workers C when workers' salaries are $\{s_m\}_{m\in M}$ is $\Pi^n(C;s) = v^n(C) - \sum_{m\in C} s_m$. We often abbreviate the tuple (N, M, v, b) to (v, b) as the sets of workers and firms are implicitly encoded in (v, b).

For any two disjoint sets of employees, C and D, we denote by $v(D|C) = v(D \cup C) - v(C)$ the marginal productivity of D given C and we also abuse notation and write m to denote the singleton set $\{m\}$ as well (hence $v(m)$ will denote the productivity of a single worker, m).

An *assignment* of workers is a partition $A = \{A^0, A^1, \ldots, A^N\}$ of the set of workers, where A^n denotes all workers employed by firm n, with A^0 implying the set of unemployed workers.

An *allocation* is a pair (A, s) where A is an assignment of workers and $s \in \Re^M_+$ is a vector of salaries. Such an allocation implies that any employee in $m \in A^n$ works for firm n at a salary s_m, whenever $n > 0$ and $m \in A^0$ implies that m is unemployed and receives no salary.

Definition 1. An allocation (A, s) is *individually rational (IR)* if (1) $v^n(A^n) - \sum_{m\in A^n} s_m \geq 0 \; \forall n \in N$; and (2) $s_m \geq b^n_m$ for all $n \in N$ and $m \in A^n$.

The first part of this definition requires that each firm has a positive net product and the second part requires that each employed worker is paid her minimal required salary.

2.1 Stability and Job Security

The central solution concept we adopt is that of stability. However our notion of stability is the central innovation of our work and is weaker than the standard stability notions in two-sided markets. The stability notion we introduce is inspired by markets where job security is guaranteed by regulatory means. In particular, we consider the following simple yet somewhat extreme assertion - once a worker is employed by a firm for a certain salary only the worker can decide to quit whereas the firm cannot lower the salary nor can it fire the worker. Thus, the stability notion we introduce is an adaptation of the standard notion of stability to such regulatory restrictions. Formally:

Definition 2. Given an allocation (A, s), we say that a coalition composed of a single firm, n, and a subset of workers $C \subset M \setminus A^n$, is a *blocking coalition* if there exists a vector of salaries, $\hat{s} \in \Re^M_+$, such that:

- $u_m(n, \hat{s}_m) \geq u_m(k, s_m) \;\; \forall k \in N, m \in A^k \cap C$ (workers in C are better off by working for n),
- $v^n(C|A^n) \geq \sum_{m\in C} \hat{s}_m$ (firm n increases profits),

with at least one of the inequalities being strict.

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which turns out to be central to our results. On the other hand this simple notion is sufficient to account for non-salary related benefits of workers via the component of a minimal salary (b in our model) which is dependant of the specific worker and the specific firm.

Definition 3. An allocation (A, s) is called *Job Security stable (JS-stable)* if it is IR and there exist no blocking coalitions.

In words, the requirement for JS-stability, beyond IR, is that there exists no firm and no set of agents currently not working for such firm such that the firm can offer better working terms for these agents (first set of inequalities) while maintaining its current set of workers and increasing its profits (second inequality). This is a weaker notion than the core allocation defined by Kelso and Crawford (1982). While Kelso and Crawford require that an allocation be immune to a deviation by a coalition of workers and a firm where such workers may (partly) replace the firm's current working force, our notion ignores this possibility. As stated previously, we make the assumption that such an option is infeasible from the outset for regulatory reasons as job security laws are in place and a firm may not unilaterally let go any of its current workers.

Realistically speaking, the regulatory environment we model allows for an extreme notion of job security. This, intuitively, could result in inefficient allocations. However, as we demonstrate in this work, in-spite of our modeling choice efficiency still prevails. This may suggest that even weaker forms of regulation designed for job security do not necessarily contradict efficiency.

2.2 AFS production functions

Throughout the paper we assume a certain structure on the production technology of each of the firms. To define this structure we recall the following definition from cooperative game theory: For any $C \subseteq M$, a vector of non-negative weights $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ is a *"fractional cover"* of C if for any $m \in C$, $\sum_{\{D \subseteq C: m \in D\}} \lambda_D = 1$.[8]

Definition 4. A firm's production function v is *Fractionally Sub-additive on* $C \subseteq M$ if for any fractional cover $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ of C, $v(C) \leq \sum_{D \subseteq C, D \neq \emptyset} \lambda_D v(D)$.

Definition 5. A firm's production function v is *Fractionally Sub-additive*, denoted $v \in FS$, if for any $C \subseteq M$, v is Fractionally Sub-additive on C.[9]

Definition 6. A vector of salaries, s, is called a *supporting salary vector* for the production function v and a subset of workers $C \subset M$ if (1) $\sum_{m \in C} s_m = v(C)$; and (2) For any $D \subset C$, $\sum_{m \in D} s_m \leq v(D)$.[10]

A well known result relating these two concepts is the Bondareva-Shapley Theorem:

[8] The term used in cooperative game theory is "balanced collection of weights", see Osborne and Rubinstein (1994).

[9] Originally, a similar notion was introduced by Bondareva (1963) and Shapley (1967) in the context of value functions for cooperative games. However, Bondareva and Shapley actually take interest in the reversed inequalities and refer to such value functions as *balanced*.

[10] Originally, Bondareva (1963) and Shapley (1967) consider the case where these inequalities are reversed and refer to a collection of such vectors as the *core* of a cooperative game

Theorem 1 (**Bondareva-Shapley Theorem).** *v is* Fractionally Sub-additive on $C \subseteq M$ *if and only if there exists a non-negative supporting vector of salaries for v an C.*[11]

We expand the class of production functions beyond FS as follows:

Definition 7. A firm's production function v is *Almost Fractionally Sub-additive*, denoted $v \in AFS$, if:

1. For any $C \subset M$ (excluding $C = M$) v is Fractionally Sub-additive on C; and
2. $v(M) \leq \frac{\sum_{m \in M} v(M \setminus m)}{|M|-1}$.

Clearly $FS \subset AFS$.

Throughout the paper we focus on this new class of production functions. This is a methodological leap in comparison with the domains of production functions studied in the literature so far which (almost) entirely focuses on production function that exhibit substitutability. Most of the literature, in fact, uses the gross-substitutes assumption introduced by Kelso and Crawford (1982). In contrast, almost fractionally sub-additive production function may exhibit complementarities, as demonstrated in the following example:

Example 1. Assume there are 3 workers, denoted a, b, c and let the production function u be defined by: $u(a) = u(b) = u(c) = 3$, $u(\{a, b\}) = u(\{a, c\}) = 6$, $u(\{b, c\}) = 4$, $u(\{a, b, c\}) = 8$. We leave it to the reader to verify that $u \in AFS$ (but not in FS). Note that the worker a and the pair $\{b, c\}$ are complementarities.

Note that the complementarity displayed in of Example 1 is possible as we do not require the fractional sub-additivity to hold on the full set of workers but only on strict subsets.

Lemma 1. *$v \in AFS \implies \sum_{m \in M} v(m|M \setminus m) \leq v(M) \leq \sum_{m \in M} v(m)$.*

Proof. To prove the left inequality note that:

$$\sum_{m \in M} v(m|M \setminus m) = \sum_{m \in M} v(M) - v(M \setminus m) = |M| \cdot v(M) - \sum_{m \in M} v(M \setminus m) \leq |M| \cdot v(M) - (|M|-1) \cdot v(M),$$

where the last inequality follows from the definition of AFS. Thus, $\sum_{m \in M} v(m|M \setminus m) \leq v(M)$.

To prove the right inequality we proceed by induction on $|M|$. The base, $|M| = 1$, is trivial. For $|M| > 1$,

$$v(M) \leq \frac{\sum_{m \in M} v(M \setminus m)}{|M| - 1} \leq \frac{\sum_{m \in M} \sum_{k \in M \setminus m} v(k)}{|M| - 1} = \sum_{k \in M} v(k),$$

where the second inequality follows from the induction hypothesis. □

[11] Bondareva and Shapley actually prove that the core is not empty if and only if the value function is balanced, which is equivalent to the stated theorem.

2.3 On a hierarchy of domains of production functions

In this section we remind the reader of additional domains of production functions which have been studied in the literature and we shed some light on how *FS* and *AFS* fit in the larger picture. This section is informal and provides no original definitions nor results. The reader can look up the formal definitions in various papers and in particular in Lehmann et al. (2006).

The most restricted class is that of Gross Substitutes, denoted *GS*. A production function is in *GS* if, given a vector of salaries worker m is chosen by the firm then an increase of salaries by workers other than m will still result in choosing to employ m.

A production function v is called sub-modular if it exhibits decreasing productivity. More formally, for every two sets of workers $S \subset T$ and for any worker element $x \notin T$, $v(x|T) \leq v(x|S)$. We denote by *SM* the set of submodular production functions.

Lehmann et al. (2006) show that $GS \subset SM \subset FS$ Moreover, they prove that GS is a zero measure set amongst all valuations with decreasing marginal values. As $FS \subset AFS$ we conclude that GS forms a zero measure subset with the set AFS.

2.4 Efficiency

The *efficiency level* of an assignment A is $P(A) = \sum_n v^n(A^n) - \sum_{m \in A^n} b^n_m$ (recall that $v^n(\cdot)$ and b^n_m are all measured in salary units). An assignment is *efficient* if it maximizes the efficiency, over all possible assignments.

3 Salary Driven Workers

Before we state our results we consider a variation of a labor market which we refer to as labor markets with *salary driven* workers. We use the notion of 'salary-driven' to emphasize that workers do not care about any aspect of their job, except for their salary. This manifests itself by setting, for each worker, a constant minimal wage across all firms. In fact, in what follows we set this constants to be zero for all workers. Thus, in a 'salary driven' labor market we formally assume $b^n_m = 0 \; \forall m \in M, n \in N$.

For labor markets with salary driven workers the notions of individual rationality and JS-stability become simpler:

Lemma 2. *Let* $b^n_m = 0 \; \forall m \in M, n \in N$. *An allocation* (A, s) *is* individually rational (IR) *if and only if* $v^n(A^n) - \sum_{m \in A^n} s_m \geq 0 \; \forall n \in N$.

The proof is straightforward and therefore omitted.

Lemma 3. *Let* (A, s) *be an allocation is a salary driven market* $(v, 0)$. *Then* (n, C) *is a* blocking coalition *if and only if* $v^n(C|A^n) > \sum_{m \in C} s_m$.

Proof. A sufficient condition: Clearly if $v^n(C|A^n) > \sum_{m \in C} s_m$ then by setting $\hat{s} = s$ the coalition (n, C) is a blocking coalition.

A necessary condition: Assume now that (n, C) is a blocking coalition and so there exits some vector of salaries $\hat{s}$ such that $v^n(C|A^n) \geq \sum_{m \in C} \hat{s}_m$ and $\hat{s}_m \geq s_m \quad \forall m \in C$, with one of the inequalities being strict. Assume the strict inequality is $v^n(C|A^n) > \sum_{m \in C} \hat{s}_m$ then clearly $v^n(C|A^n) > \sum_{m \in C} s_m$ as well and we are done. If, on the other hand, $v^n(C|A^n) = \sum_{m \in C} \hat{s}_m$ but for some $\hat{m} \in C$, $\hat{s}_{\hat{m}} > s_{\hat{m}}$ then once again we have $v^n(C|A^n) > \sum_{m \in C} s_m$ and we are done. □

As before an allocation (A, s) is called *Job Security stable (JS-stable)* if it is IR and there exist no blocking coalitions.[12]

3.1 labor markets, without loss of generality, have salary driven workers

In this section we construct tools that will enable us to prove results about labor markets by restricting attention to labor markets that have salary driven workers. Let (v, b) be a labor market. We denote by $(v - b, 0)$ be a labor market with salary driven agents and production functions $(v - b)^n(B) = v^n(B) - \sum_{m \in B} b^n_m$. Similarly let (A, s) be some allocation then $s - b$ is the following vector of salaries: If $m \in A^k$ then $(s - b)_m = s_m - b^k_m$.

Lemma 4. *Let (v, b) be a labor market. $v^n \in AFS$ if and only if $(v - b)^n \in AFS$.*

Proof. We prove the first direction and assume $v^n \in AFS$. To prove that $(v - b)^n \in AFS$ we need to show 2 things:

1. For any $C \subset M, C \neq M$, $v - b$ is fractionally Sub-additive on C. Indeed, let $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ be a fractional cover of C and so $\sum_{m \in C} b_m = \sum_{D \subseteq C, D \neq \emptyset} \lambda_D \sum_{m \in D} b_m$. Consequently:

$$(v-b)(C) = v(C) - \sum_{m \in C} b_m \leq \sum_{D \subseteq C, D \neq \emptyset} \lambda_D v(D) - \sum_{D \subseteq C, D \neq \emptyset} \lambda_D \sum_{m \in D} b_m = \sum_{D \subseteq C, D \neq \emptyset} \lambda_D (v-b)(D).$$

2. $(v - b)(M) \leq \frac{\sum_{m \in M} (v-b)(M \setminus m)}{|M|-1}$. Indeed

$$(v - b)(M) = v(m) - \sum_{m} b_m \leq \frac{\sum_{m \in M} v(M \setminus m)}{|M| - 1} - \frac{\sum_m \sum_{k \neq m} b_k}{|M| - 1} = \frac{\sum_{m \in M} (v - b)(M \setminus m)}{|M| - 1}.$$

The opposite direction of the proof is similar and hence omitted. □

Let $P_{(v,b)}(A)$ be the efficiency level of the assignment A for the labor market (v, b).

[12] The model of salary driven workers may be viewed as a combinatorial auction model with a seller who owns goods (the "workers") and buyers (the "firms") who have valuations for subsets of goods and compete for these goods. The interpretation of IR and efficiency in this model is straightforward, however thew notion of a 'blocking coalition' and hence the notion of JS-stability form a technical relaxation of the notion of a core for which the authors have compelling explanation.

Lemma 5. $P_{(v,b)}(A) = P_{(v-b,0)}(A)$.

Proof. This is quite straightforward:

$$P_{(v,b)}(A) = \sum_n (v^n(A^n) - \sum_{m \in A^n} b^n_m) = \sum_n (v-b)^n(A^n) = P_{(v-b,0)}(A).$$

□

Lemma 6. *A is an efficient assignment for $(v-b, 0)$ if and only if it is an efficient assignment for (v, b).*

Proof. This follows directly from Lemma 5. □

Lemma 7. *The allocation (A, s) is an (IR) allocation for the labor market (v, b) if and only if $(A, s-b)$ is an (IR) allocation for $(v-b, 0)$.*

Proof. Let (A, s) be an (IR) allocation for (v, b). Then, for each firm n, $v^n(A^n) \geq \sum_{m \in A^n} s_m$ which can be rewritten as $(v-b)^n(A^n) \geq \sum_{m \in A^n}(s_m - b^n_m) = \sum_{m \in A^n}(s-b)_m$. In addition, for each worker, m, $s_m \geq b^n_m$, where $m \in A^n$. Equivalently , $(s-b)_m \geq 0$ which means that $(A, s-b)$ is (IR) in the labor market $(v-b, 0)$.

The proof of the opposite direction is similar and hence omitted. □

Lemma 8. *The coalition (n, C) is a blocking coalition for the allocation (A, s) in the labor market (v, b) if and only if it is a blocking coalition for the allocation $(A, s-b)$ in the labor market $(v-b, 0)$.*

Proof. Assume that (n, C) is a blocking coalition for the allocation (A, s) in the labor market (v, b) . Then there exists some vector of salaries $\{\hat{s}_m\}_{m \in C}$ such that:

- $\hat{s}_m - b^n_m \geq s_m - b^k_m$ for all k and for all $m \in C \cap A^k$,
- $v^n(C|A^n) \geq \sum_{m \in C} \hat{s}_m$, implying $(v-b)^n(C|A^n) \geq \sum_{m \in C} \hat{s}_m - b^n_m$

with at least one of the inequalities being strict.

Let us set $\bar{s}_m = \hat{s}_m - b^n_m, \ \forall m \in C$. The above system of inequalities is equivalent to:

- $\bar{s}_m \geq s_m - b^k_m = (s-b)_m \ \ \forall k$ and $m \in A^k \cap C$,
- $(v-b)^n(C|A^n) \geq \sum_{m \in C} \bar{s}_m$,

with at least one of the inequalities being strict, implying the desired conclusion.

The proof of the opposite direction is similar and hence omitted. □

Lemma 9. *The allocation (A, s) is a JS-stable allocation for (v, b) if and only if the allocation $(A, s-b)$ is a JS-stable allocation for $(v-b, 0)$.*

Proof. This is a direct consequence of Lemmas 7 and 8. □

4 Results

We provide 4 results. The first two connect JS-stability with efficiency and can be viewed as analogs for the first and second social welfare. In particular, we show that whenever production functions are in *AFS* efficient outcomes are JS-stable. In our third result we show that one cannot obtain such a results beyond such production functions. Our final result, makes a connection between the notion of JS-stability, inspired by cooperative game theory, and a Nash equilibrium of a natural auction-like non-cooperative game played among the firms.

4.1 A $\frac{1}{2}$-First Welfare Theorem

As one can expect, JS-stability does not guarantee efficiency. On the other hand the inefficiency of any JS-stable outcome in bounded:

Theorem 2. *If (A, s) is a JS-stable allocation and $\bar{A}$ is an efficient assignment then , $P(A) \geq \frac{1}{2}P(\bar{A})$.*

Note that this result assumes no restrictions on production technologies.

Proof. We first prove our result for labor markets with salary driven workers, denoted $(v, 0)$. Indeed, for every firm n we have $v^n(\bar{A}^n \setminus A^n | A^n) \leq \sum_{m\in\bar{A}^n\setminus A^n} s_m$. Thus, we have

$$v^n(\bar{A}^n) \leq v^n(\bar{A}^n \cup A^n) \leq \sum_{m\in\bar{A}^n\setminus A^n} s_m + v^n(A^n)$$

Therefore

$$\sum_{i=1}^{n} v^n(\bar{A}^n) \leq \sum_{i=1}^{n}\left(\sum_{m\in\bar{A}^n\setminus A^n} s_m + v^n(A^n)\right) \leq \sum_{i=1}^{n}\left(\sum_{m\in\bar{A}^n} s_m + v^n(A^n)\right) \leq$$

$$\leq \sum_{m\in M} s_m + \sum_{i=1}^{n} v^n(A^n) = \sum_{i=1}^{n}\sum_{m\in A^n} s_m + \sum_{i=1}^{n} v^n(A^n) \leq 2\sum_{i=1}^{n} v^n(A^n),$$

where the last inequality follows from (IR) of the assignment $A = (A^n)_{n\in N}$. This proves the claim for labor markets with salary driven workers.

Now let (A, s) be a JS-stable allocation for an arbitrary labor market (v, b) and let $\bar{A}$ be an efficient assignment for (v, b). Therefore, $(A, s - b)$ is a JS-stable allocation for $(v - b, 0)$ (Lemma 9) and $\bar{A}$ is efficient for $(v - b, 0)$ (Lemma 6). Now:

$$P_{(v,b)}(A) = P_{(v-b,0)}(A) \geq \frac{1}{2}P_{(v-b,0)}(\bar{A}) = \frac{1}{2}P_{(v,b)}(\bar{A}),$$

where the left and right equalities follow from Lemma 5 and the inequality follows from the first part of the proof. $\square$

This bound on the efficiency loss is tight as suggested by the following example:

Example 2. Consider a labor market with four salary-driven workers a, b, c, d and two firms with production functions, v_1, v_2, defined as follows. For any non-empty subset of workers $S \subseteq \{a, b, c, d\}$, $v_1(S)$ is 1 unless $\{a, c\} \subseteq S$, in which case $v_1(S)$ is 2; similarly, $v_2(S)$ is 1 unless $\{b, d\} \subseteq S$, in which case $v_2(S)$ is 2.

Let $S_1 = \{a, b\}$, $S_2 = \{c, d\}$ and set wages $p(b) = p(c) = 1$, $p(a) = p(d) = 0$. It is straightforward to verify this is a JS-stable allocation and the social welfare is 2, whereas the social welfare of the efficient assignment is 4.

4.2 A Second Welfare Theorem

Theorem 3. *Let (v, b) be a labor market. If $v^n \in AFS$ for all $n \in N$ then for any efficient assignment A there is a salary vector s, such that (A, s) is a JS-stable allocation.*

Note, in particular that the existence of a JS-stable outcome is guaranteed under the conditions of Theorem 3

Proof. We begin by proving our result for an arbitrary salary driven job $(v, 0)$, with production functions in AFS.
Case 1: No efficient assignment assigns all workers to a single firm: Therefore if $A = (A^1, \cdots, A^n)$ is some efficient assignment then $A^k \neq M$ for any firm k. Thus, we can apply Theorem 1 (which is our version of the Bondareva Shapley theorem) and conclude that for each $k \in N$, there exists a supporting vector of salaries, $\{s^k_m\}_{m\in A^k}$, for (v^k, A^k). For any $m \in M$ let $n(m)$ denote the firm for which $m \in A^{n(m)}$ and set $s_m = s^{n(m)}_m$. We show that the allocation (A, s) is JS-stable. IR follows immediately from the definition of a supporting vector of salaries. To finish our proof we must show that an arbitrary coalition, (n, B), where $B \subset M \setminus A^n$, cannot be a blocking coalition. Denote $R^k = A^k \cap B$. As A is efficient $v^n(A^n \cup B) + \sum_{k\neq n} v^k(A^k \setminus R^k) \leq \sum_{k\in N} v^k(A^k)$. Therefore $v^n(A^n) + v^n(B|A^n) \leq \sum_{k\in N} v^k(A^k) - \sum_{k\neq n} v^k(A^k \setminus R^k) = v^n(A^n) + \sum_{k\neq n} v^k(R^k|A^k \setminus R^k)$. As $\{s^k_m\}_{m\in A^k}$ is a vector of supporting salaries for (v^k, A^k) we have $v^n(B|A^n) \leq \sum_{k\neq n} v^k(R^k|A^k\setminus R^k) \leq \sum_{k\neq n}\sum_{m\in R^k} s^k_m = \sum_{m\in B} s_m$, implying that (n, B) is not a blocking coalition.
Case 2: There is an efficient assignment that assigns all workers to firm n: Efficiency implies that for any $k \neq n$ and any $m \in M$, $v^k(m) + v^n(M \setminus m) \leq V^n(M) = v^n(m|M \setminus m) + v^n(M \setminus m)$, therefore $v^k(m) \leq v^n(m|M \setminus m)$.

Now set $s_m = v^n(m|M \setminus m)$ for every $m \in M$. We show that this yields a JS-stable allocation:

- **IR**: By Lemma 1 $v^n(M) \geq \sum_{m\in M} v^n(m|M \setminus m) = \sum_{m\in M} s_m$, and IR follows from Lemma 2.

- **No blocking coalition**: For every firm $k \neq n$ and for every subset $B \subseteq M$, we apply Lemma 1:
$$v^k(B) \leq \sum_{m \in B} v^k(m) \leq \sum_{m \in B} v^n(m|M \setminus m) = \sum_{m \in B} s_m,$$
Thus no blocking coalition follows from Lemma 3.

So far we proven our claim for a salary driven labor market. The proof for an arbitrary labor market follows from Lemmas 4, 6 and 9. □

Note that the proof of Theorem 3 is constructive and so it is suggestive of an algorithm to compute optimal JS-stable allocations.

4.3 The maximality of the set of production technologies AFS

We now turn to show that the set of production functions, AFS, is maximal with respect to the property that any efficient assignment can also be supported as a JS-stable allocation. In other words, if one of the firms has a production function that is not in AFS it could be the case that some efficient assignment is not supported by a JS-stable allocation. In fact, we will show that it could be that none of the efficient assignments are supported by a JS-stable allocation. Formally:

Theorem 4. *If $\bar{v} \notin AFS$ then there exists a labor market $(v, 0)$, where $v^1 = \bar{v}$ and for all $n > 1$ $v^n \in AFS$ and if A is an efficient assignment then for no vector of salaries s is (A, s) a JS-stable allocation of the market $(v, 0)$.*

Before proceeding to the proof we need some interim observations:

Lemma 10. *For any monotone production function v and for any fractional cover $\{\lambda_D : D \subseteq M\}$:*
$$\frac{v(M) - \sum_{D \subseteq M} \lambda_D v(D)}{\sum_{D \subseteq M} \lambda_D - 1} \leq \max\{1, (|M| - 1)v(M)\},$$
where $\frac{0}{0} = 1$.

Proof. If $\lambda_M = 1$ then for all $D \neq M$ it must be the case that $\lambda_D = 0$ and the ratio on the left hand side os the desired inequality is 1 while the right hand side is also 1 and so the claim follows.

Let $\{\lambda_D : D \subseteq M\}$ be a fractional cover of M such that $1 \neq \lambda_M > 0$. Then:
$$\frac{v(M) - \sum_{D \subseteq M} \lambda_D v(D)}{\sum_{D \subseteq M} \lambda_D - 1} = \frac{(1 - \lambda_M)v(M) - \sum_{D \subseteq M, D \neq M} \lambda_D v(D)}{\sum_{D \subseteq M, D \neq M} \lambda_D - (1 - \lambda_M)} =$$
$$\frac{v(M) - \sum_{D \subseteq M, D \neq M} \frac{\lambda_D}{1-\lambda_M} v(D)}{\sum_{D \subseteq M, D \neq M} \frac{\lambda_D}{1-\lambda_M} - 1}.$$

As $\left\{\frac{\lambda_D}{1-\lambda_M}\right\}_{D\subseteq M, D\neq M}$ is a fractional cover of M it is enough to show that

$$\sup\left\{\frac{v(M)-\sum_{D\subseteq M}\lambda_D v(D)}{\sum_{D\subseteq M}\lambda_D - 1} \mid \{\lambda_D\}_{D\subseteq M}, \lambda_M = 0\right\} \leq (|M|-1)v(M).$$

Now let λ_D be a covering of M such that $\lambda_M = 0$. Therefore $\lambda_D > 0$ implies that $M \setminus D \neq \emptyset$. Let χ denote the indicator function.

$$|M|\sum_D \lambda_D = \sum_{m\in M}\sum_D \lambda_D \chi(m\in D) + \sum_{m\in M}\sum_D \lambda_D\chi(m\in M\setminus D) =$$
$$= |M| + \sum_D \lambda_D \sum_{m\in M}\chi(m\in M\setminus D) \geq |M| + \sum_D \lambda_D,$$

where the last inequality hinges on $\lambda_D > 0 \implies M\setminus D \neq \emptyset$.

Therefore $\sum_D \lambda_D \geq \frac{|M|}{|M|-1}$ and so $\frac{1}{\sum_D \lambda_D - 1} \leq |M|-1$. This implies that $\frac{v(M)-\sum_{D\subseteq M}\lambda_D v(D)}{\sum_{D\subseteq M}\lambda_D - 1} \leq (|M|-1)v(M)$. □

For any valuation v and a positive number r let $v+r$ be the valuation defined as follows: $(v+r)(D) = v(D)+r,\ \forall D\subseteq M$.

Lemma 11. *For any monotone valuation v, there exists some positive number R such that for any $r \geq R,\ \ v+r \in FS$*

Proof. Let $R = \max\{1, (|M|-1)v(M)\}$ and assume that for some $r > R$, $v+r \notin FS$. This implies that there exists some $T\subseteq M$ and a fractional cover, $\{\lambda_D : D\subseteq T\}$ of T, such that $\sum_D \lambda_D(v+r)(D) < (v+r)(T)$. And so $\sum_D \lambda_D v(D) + r\sum_D \lambda_D < v(T)+r$, or alternatively $\frac{v(T)-\sum_D \lambda_D v(D)}{\sum_D \lambda_D - 1} > r \geq \max\{1,(|M|-1)v(M)\} \geq \max\{1,(|T|-1)v(T)\}$, which in turn, contradicts the previous lemma applied to the set T. □

For any $T\subset M$ let $v|_T(\cdot)$ denote the restriction of $v(\cdot)$ to T.

Lemma 12. *If $v\notin FS$ and for all strict subsets, $T\subset M$, $v|_T \in FS$ then for all strict subsets $T\subset M$, $v(T) < v(M)$.*

Proof. As $v\notin FS$ there exists some fractional cover $\{\lambda_{\bar{T}}\}$ of M such that $v(M) > \sum_{\bar{T}}\lambda_{\bar{T}} v(\bar{T})$. Assume the claim is wrong and for some $T\subset M$, $v(T) = v(M)$.

For any $D\subset T$ let $\beta_D = \sum_{\{\bar{D}:\bar{D}\cap T = D\}}\lambda_{\bar{D}}$. Note that for any $m\in T$:

$$\sum_{\{D\subseteq T: m\in D\}}\beta_D = \sum_{\{D\subseteq T:m\in D\}}\sum_{\{\bar{D}:\bar{D}\cap T = D\}}\lambda_{\bar{D}} = \sum_{\{\bar{D}:m\in\bar{D}\}}\lambda_{\bar{D}}.$$

Thus, $\{\beta_D\}$ is a fractional cover of T and therefore, as $v|_T \in FS$, the following holds:

$$\sum_{D\subset T} \beta_D v(D) = \sum_{D\subset T} \beta_D v|_T(D) \geq v|_T(T) = v(T).$$

From this we can deduce:

$$v(M) > \sum_{\bar{T}} \lambda_{\bar{T}} v(\bar{T}) \geq \sum_{\bar{T}} \lambda_{\bar{T}} v(\bar{T}\cap T) = \sum_{D\subset T} \sum_{\{\bar{T}:\bar{T}\cap T = D\}} \lambda_{\bar{T}} v(\bar{T}\cap T) =$$

$$= \sum_{D\subset T} \sum_{\{\bar{T}:\bar{T}\cap T = D\}} \lambda_{\bar{T}} v(D) = \sum_{D\subset T} \beta_D v(D) \geq v(T) = v(M),$$

and so we reach a contradiction. □

We say that the production function v is a unit demand production function if $v(B) = \max_{m\in B} v(m)$ for any $B \subseteq M$. Thus, to define a unit demand production function it suffices to define the set of numbers $\{v(m) : m \in M\}$. It is easy to verify that a unit demand production function is in FS and hence also in AFS.

Proof. (Theorem 4)

We split the proof into two cases:

Case 1: For some $B \subseteq M,\ \bar{v}(B) > \sum_{m\in B} \frac{\bar{v}(B\setminus m)}{|B|-1}$.

In this case

$$\sum_{m\in B} \bar{v}(m|B\setminus m) = \sum_{m\in B} \bar{v}(B) - \bar{v}(B\setminus m) = |B|\cdot \bar{v}(B) - \sum_{m\in B} \bar{v}(B\setminus m) > |B|\cdot \bar{v}(B) - (|B|-1)\cdot \bar{v}(B),$$

Therefore $\sum_{m\in B} \bar{v}(m|B\setminus m) > \bar{v}(B)$. We construct the following tuple of unit-demand production functions. For every worker $m \in M\setminus B$ we have two production functions $v_m^{(1)} = v_m^{(2)}$ such that $v_m^{(i)}(m) = \bar{v}(M)+1$ and $v_m^{(i)}(k) = 0$ for any worker $k \neq m$. Additionally define a unit-demand production function v^B as follows. Choose a small enough $\epsilon > 0$ such that (i) $\sum_{m\in B}(\bar{v}(m|B\setminus m) - \epsilon) > \bar{v}(B)$, and (ii) $\forall m \in B$ such that $\bar{v}(m|B\setminus m) > 0$, $\epsilon < \bar{v}(m|B\setminus m)$. Then define

$$v^B(m) = \begin{cases} \max(0, \bar{v}(m|B\setminus m) - \epsilon) & m \in B \\ 0 & m \notin B \end{cases}$$

We show that there exists no JS-stable allocation for the labor market $((\bar{v}, v^B, \{v_m^{(1)}, v_m^{(2)}\}_{m\in M\setminus B}), 0)$. Suppose by contradiction that there exists a JS-stable allocation $((A^{\bar{v}}, A^B, \{A_m^1, A_m^2\}_{m\in M\setminus B}), s)$. Every worker $m \in M\setminus B$ must be allocated to either the firm with production function $v_m^{(1)}$ or $v_m^{(2)}$ and its salary must be $v_m^{(1)}(m)$. As a result,$A^{\bar{v}} \cup A^B \subseteq B$ and if $k \in B \cap A_m^i$ for some $i = 1, 2$ and some $m \in M \setminus B$ then $s_k = 0$ for otherwise the firm with production function $v_m^{(i)}$ violates IR. This implies that $\sum_{m \in A^{\bar{v}} \cup A^B} s_m = \sum_{m \in B} s_m$.

Once again we split to two cases:

Case 1a: $v^B(A^B) = 0$. In this case by the definition of v^B for any $m \in A^B$, $v^B(m) = 0$ and so $\bar{v}(m|B \setminus m) - \epsilon \leq 0$. In addition, as the allocation is JS-stable $\sum_{m \in A^B} s_m$ must be 0. Therefore,

$$\sum_{m \in B \setminus A^B} s_m = \sum_{m \in B} s_m = \sum_{m \in A^{\bar{v}}} s_m \leq \bar{v}(A^{\bar{v}}) \leq \bar{v}(B) < \sum_{m \in B} (\bar{v}(m|B \backslash m) - \epsilon) \leq \sum_{m \in B \setminus A^B} (\bar{v}(m|B \backslash m) - \epsilon).$$

Thus, there exists a worker $m \in B \setminus A^B$ with $s_m < \bar{v}(m|B \setminus m) - \epsilon \leq v^B(m)$, thus contradicting JS-stability (recall Lemma 3).

Case 1b: $v^B(A^B) > 0$. Let $m^* = \text{argmax}_{m \in A^B} v^B(m)$ then clearly $v^B(A^B) = v(m^*)$. From the definition of v^B we have $\bar{v}(m^*|B \setminus m^*) \geq v^B(m^*) + \epsilon$. Since $\sum_{m \in B \setminus A^{\bar{v}}} s_m = \sum_{m \in A^B} s_m \leq v^B(A^B) = v^B(m^*)$ we may conclude that $\sum_{m \in B \setminus A^{\bar{v}}} s_m < \bar{v}(m^*|B \setminus m^*)$ and so:

$$\begin{aligned}
\bar{v}(B \setminus A^{\bar{v}}|A^{\bar{v}}) - \sum_{m \in B \setminus A^{\bar{v}}} s_m &> \bar{v}(B \setminus A^{\bar{v}}|A^{\bar{v}}) - \bar{v}(m^*|B \setminus m^*) \\
&= (\bar{v}(B) - \bar{v}(A^{\bar{v}})) - (\bar{v}(B) - \bar{v}(B \setminus m^*)) \\
&= \bar{v}(B \setminus m^*) - \bar{v}(A^{\bar{v}}) \geq 0,
\end{aligned}$$

Recall that $A^{\bar{v}} \cup A^B \subseteq B$ and so the last inequality follows since $A^{\bar{v}} \subseteq B \setminus A^B \subseteq B \setminus m^*$. This inequality implies that the allocation is not JS-stable (recall Lemma 3).

Case 2: For all $B \subset M$, $\bar{v}(B) \leq \sum_{m \in B} \frac{\bar{v}(B \backslash m)}{|B|-1}$. As $\bar{v} \notin AFS$ there exists some strict subset $T \subset M$ such that $\bar{v}|_T \notin FS$. In particular, let T be a minimal such subset, namely any strict subset of T is in FS. By Lemma 12 for any T' that is a strict subset of T, $\bar{v}(T') < \bar{v}(T)$. In particular we may choose $\bar{\epsilon} > 0$ be such that for any T' that is a strict subset of T, $\bar{v}(T') + \bar{\epsilon} < \bar{v}(T)$.

For any $\bar{\epsilon} > \epsilon > 0$ we define the valuation u^ϵ on M as follows: $u^\epsilon(D) = r - \bar{v}(D^c)\ \forall D \neq T^c$ and $u^\epsilon(T^c) = r - \bar{v}(T) + \epsilon$, where $r = r(\epsilon)$ is large enough to guarantee that $u^\epsilon \in FS$ (recall Lemma 11).[13] Monotonicity of u^ϵ is straightforward from the construction and the choice of ϵ.

Allocating T to the firm with production function $\bar{v}$ and T^c to the agent with production function u^ϵ is the unique optimal allocation. Note that it generates a social welfare of $r + \epsilon$ whereas any other allocation generates r.

Assume the theorem is wrong and that for any ϵ the unique optimal assignment of $(\bar{v}, u^\epsilon)$ can be supported by a JS-stable allocation $((T, T^c), s^\epsilon)$. By IR $\sum_{m \in T} s_m^\epsilon \leq \bar{v}(T)$, however by increasing the salary of some single worker in T we can assume, without loss of generality, that $\sum_{m \in T} s_m^\epsilon = \bar{v}(T)$.

[13] The set $D^c = M \setminus D$ denotes the complementary set of D in M.

For any $D \subseteq T$, JS-stability implies

$$\sum_{m \in D} s_m^\epsilon \geq u^\epsilon(D|T^c) = u^\epsilon(D \cup T^c) - u^\epsilon(T^c) = \bar{v}(T) - \bar{v}(T \setminus D) - \epsilon = \sum_{m \in T} s_m^\epsilon - \bar{v}(T \setminus D) - \epsilon.$$

Therefore, for any $D \subseteq T$, $\sum_{m \in T \setminus D} s_m^\epsilon \leq \bar{v}(T \setminus D) + \epsilon$. This can be equivalently stated as follows:

$$\sum_{m \in D} s_m^\epsilon \leq \bar{v}(D) + \epsilon \quad \forall D \subseteq T.$$

Let $\bar{\epsilon} > \epsilon_n > 0$ be decreasing sequence with $\lim_n \epsilon_n = 0$ and let s be an accumulation point of the set of salary vectors $\{s^{\epsilon_n}\}_{n=1}^\infty$. Then $\sum_{m \in T} s_m = \bar{v}(T)$ and $\sum_{m \in D} s_m \leq \bar{v}(D) \ \ \forall D \subset T$ which implies that s is a supporting vector of salaries for $\bar{v}|_T$ on the set T, contradicting the assumption that $\bar{v}|_T \notin FA$. □

Theorem 4 shows that *AFS* is a maximal domain of production functions such that the second welfare theorem holds. In particular if $v \in AFS$ then a JS-stable ouitcome is guaranteed to exist. However, it may not be maximal if we only require the existence of JS-stable outcomes. The following result demonstrates that mildly relaxing the requirements underlying *AFS* could yield such a maximal set. To state our next result we need the following definition:

Definition 8. A valuation s called *symmetrically fractionally sub-additive* if for any $B \subseteq M$ with $|B| \geq 2$, $v(B) \leq \frac{1}{|B|-1} \sum_{x \in B} v(B \setminus x)$. Let *SFS* denote the set of all symmetric fractionally sub-additive valuations.

Clearly $AFS \subset SFS$ and in addition:

Lemma 13. $v \in SFS \implies \sum_{m \in M} v(m|M \setminus m) \leq v(M) \leq \sum_{m \in M} v(m)$.

Note that this result is similar to Lemma 1 and in fact the proof of Lemma 1 applies verbatim to this lemma as well and so it is omitted.

With this at hand we turn to the following theorem which sheds some light on the structure of maximal domains which guarantee the existence of JS-stable allocations:

Theorem 5. *If $u \notin SFS$ then there exist unit-demand valuations $v_1, ..., v_n$ such that the salary driven labor market with $n+1$ workers, $((v_1, ..., v_n, u), 0))$, does not admit a JS-stable allocation.*

Proof. Since $u \notin SFS$, Lemma 13 implies that there exists $B \subseteq M$ such that $\sum_{x \in B} u(x|B \setminus x) > u(B)$. We construct the following tuple of unit-demand valuations. For every worker $x \in M \setminus B$ we have two unit-demand valuations $v_x^{(1)} = v_x^{(2)}$ such that $v_x^{(i)}(x) = u(M) + 1$ and $v_x^{(i)}(y) = 0$ for any worker $y \neq x$. Additionally define a unit-demand valuation v_B as follows. Choose a small enough $\epsilon > 0$ such that (i) $\sum_{x \in B}(u(x|B \setminus x) - \epsilon) > u(B)$, and (ii) $\forall x \in B$ such that $u(x|B \setminus x) > 0$,

$\epsilon < u(x|B \setminus x)$. Then define

$$v_B(x) = \begin{cases} \max(0, u(x|B \setminus x) - \epsilon) & x \in B \\ 0 & x \notin B \end{cases}$$

We show that that there does not exist a JS-stable allocation for this labor market. Note that in every possible JS-stable allocation in this labor market, every worker $x \in M \setminus B$ must be allocated to either the firm with valuation $v_x^{(1)}$ or $v_x^{(2)}$ and its salary must be $v_x^{(1)}(x)$. As a result, note that if in some JS-stable allocation a firm with valuation $v_x^{(i)}$ is being allocated some worker $y \in B$, y's salary must be zero. Suppose by contradiction that there exists a JS-stable allocation with salaries p and in which the firm with valuation u is allocated a set of workers T_u, the firm with valuation v_S is allocated a set of workers T_v, and $T_u \cup T_v \subseteq B$.

If $T_v = \emptyset$ or $v_B(T_v) = 0$ (in which case $p(T_v)$ is 0), then we have $p(B) = \sum_{x \in T_u} p_x \leq u(T_u) \leq u(B) < \sum_{x \in B}(u(x|B \setminus x) - \epsilon)$. Thus, there exists a worker $x \in B \setminus T_v$ with $p_x < u(x|B \setminus x) - \epsilon \leq v_B(x)$. Since in this case player the firm with valuation v_B will desire such a worker x, this cannot be a JS-stable allocation.

Otherwise, $v_B(T_v) > 0$. Let $x^* = \text{argmax}_{x \in T_v} v_B(x)$, then $v_B(x^*) = u(x^*|B \setminus x^*) - \epsilon$. Since $p(B \setminus T_u) = p(T_v) \leq v_B(x^*)$ we have,

$$\begin{aligned} u(B \setminus T_u|T_u) - p(B \setminus T_u) &> u(B \setminus T_u|T_u) - u(x^*|B \setminus x^*) \\ &= (u(B) - u(T_u)) - (u(B) - u(B \setminus x^*)) \\ &= u(B \setminus x^*) - u(T_u) \geq 0, \end{aligned}$$

where the last inequality follows since $T_u \subseteq B \setminus x^*$. Once again this contradicts the assumption that the allocation is JS-stable. □

4.4 JS-stability as an outcome of a decentralized mechanism

A labor market, (v, b), naturally induces the following complete information normal-form game played among the firms. Each firm proposes a vector of salaries, one for each worker (and firm 0 proposes the vector 0). Each worker is then assigned to the firm that proposed the best salary, while receiving a salary that would make him indifferent between his best offer and his second best offer. We refer to this game as the *Second-Price Item Bidding (SPIB)* game. Our last result shows that a JS-stable allocation is also an equilibrium outcome of the SPIB game.

Formally, each firm proposes a salary schedule $p^n = \{P_m^n\}_{m \in M}$. Given a vector of proposals $\vec{p} = (p^1, \ldots, p^N)$ and given a worker m, let $k_m = argmax_{n \in N \cup \{0\}} p_m^n - b_m^n$ and $s_m = \min\{s : s - b_m^{k_m} \geq p_m^n - b_m^n \forall n \neq k_m\}$.

Note that the existence of a pure Nash equilibrium is guaranteed. In fact, in any profile of bids where one firm proposes an "infinite" salary to all workers, while all other firms propose a minimal salary (one that makes workers indifferent between working and staying unemployed) is such an equilibrium.

In what follows we rule out such equilibria by following ideas proposed in Christodoulou et al. (2008) and Bhawalkar and Roughgarden (2011). We restrict attention to *Nash equilibria with no overbidding*, which are those Nash equilibria, $\vec{p}$, in which, for any firm n and any subset of workers $D \subseteq M$, $v^n(D) \geq \sum_{m \in D} p^n_m$. In words, firms' proposal is such that no matter which workers they are eventually assigned they do not lose. In fact we consider here a weaker restriction which we refer to as *Nash equilibria with weak no-overbidding*, which are those Nash equilibria $\vec{p}$ in which for any firm n, $v^n(D^n(\vec{p})) \geq \sum_{n \in D^n(\vec{p})} p^n_m$, where $D^n(\vec{p})$ is the set of workers assigned to firm n for the vector of proposals $\vec{p}$. In other words, the no-overbidding restriction involves only the set of workers the firm is eventually allocated.

Theorem 6. *For any labor market (v, b) there exists a pure Nash equilibrium with weak no-overbidding in the induced SPIB game if and only if there exists JS-stable allocation in the market. Moreover, the underlying transformation between the NE and the JS-stable allocation preserves the assignment of workers to firms.*

Proof. Let $\vec{p}$ be a pure Nash equilibrium with weak no-overbidding for the SPIB game induced by (v, b). We construct a JS-stable allocation in the following way: the set of workers D^n assigned to firm n is exactly $D^n(\vec{p})$ the set of workers assigned to n in the equilibrium of the SPIB game. The salary of worker $m \in M$ will be $s_m = \max_{n \in N} p^n_m$. By definition, for every firm n and every $m \in D^n$, $p^n_m = s_m$. Thus, by weak no-overbidding, $v^n(D^n) \geq \sum_{m \in D^n} s_m$, implying the individual rationality requirement. Since $\vec{p}$ is a Nash equilibrium, for any firm n and any $T \subset M \setminus D^n$, $v^n(T|D^n) \leq \sum_{m \in T} s_m$, since otherwise firm n can strictly increase utility in the SPIB game by proposing very high salary to the workers in T.

Now suppose that there exists a JS-stable allocation $(S_1, ..., S_n)$ and $(p_1, ..., p_n)$ for $(v_1, ..., v_n)$. We claim that the following bid vector $\vec{b}$ is a pure Nash equilibrium with weak no-overbidding for SPIB with $(v_1, ..., v_n)$.

$$b^j_i = \begin{cases} p_j & j \in S_i \\ 0 & j \notin S_i \end{cases}$$

Note that with this bid vector, each player i wins S_i, and pays zero. Clearly player i cannot increase her utility by changing any bid for a worker in S_i, as she pays zero for these workers. She also cannot increase utility by bidding higher on workers in some subset $T \subset \Omega \setminus S_i$, as she will have to additionally pay $\sum_{j \in T} p_j \geq v_i(T|S_i)$ for these workers, where the inequality follows from JS-stability. Thus $\vec{b}$ is indeed a Nash equilibrium. By individual rationality, $v_i(S_i) \geq \sum_{j \in S_i} p_j$, implying that $\vec{b}$ satisfies weak no-overbidding, and the claim follows. □

5 Discussion and Future Research

In this work we introduce JS-stability as a new solution concept for many-to-many matching markets. This concept is inspired by regulated labor markets where costs for firing employees are prohibitively high. It is quite straightforward to prove that any stable outcome, in the classical sense, is also JS-stable. However, there are JS-stable outcomes which are not stable. In fact, there is a large family of production functions which do not admit stable outcomes yet JS-stable outcomes not only exist but in fact support all efficient outcomes. Unfortunately, JS-stability does not always guarantee efficiency. Surprisingly, it does guarantee a (multiplicative) upper bound of 50% on efficiency loss.

5.1 Regulated labor markets

Harnessing the many-to-many matching model for studying regulation in labor markets is novel, to the best of our knowledge. Thus, our work is only a first step in a research agenda on regulation in general and JS-stability in particular that can shed light on the implications of regulatory intervention in labor markets. We highlight some natural follow-up questions which we leave for future research:

- Much of the work done by labor theorists around termination costs for workers focuses on the implications of such costs on the unemployment level. Two contradicting forces come into play. First, due to high termination costs employees will not be fired and hence unemployment should decrease. Second, at the hiring stage firms take the termination costs into account and so tend to hire less. The lion's share of the related work uses partial or general equilibrium analysis. In particular it assumes a homogeneous workforce. Our model, on the other hand, assumes heterogeneity of the workers and so may lead to conclusions that are different from those reached via the homogeneity assumption. One natural question is to compare employment levels in stable outcomes with those of JS-stable outcomes, when both exist (e.g., under gross-substitutes assumption). It should not be surprising if there are JS-stable outcomes where unemployment levels are high. However, it could be surprising to demonstrate cases where JS-stable outcomes exhibit lower unemployment rates (or to prove this cannot happen).

- As mentioned, JS-stability is inspired by prohibitive firing costs for employers. What if firing costs are moderate? What kind of a model and solution concept should this inspire? The answer could well depend on who benefits from these payments or on whether they are deadweight costs. More generally, what is the right model and solution concept to use in order to accurately capture other regulatory means designed for job protection and job security (e.g., insurance institutions such as social security)?

- Recall that some of our results refer to a cardinal notion of efficiency. For this notion to make sense we require that all utilities, for firms and for workers, are given in the same 'currency'. As a result our model assumes that firms' and workers' utilities are given in terms of money. Whereas for firms this is natural (as we identify utility with profits), for workers this is a limitation. Therefore, a study of JS-stability is called for when workers' utility functions go beyond additive-separable functions. This is particularly important if one would like to account for uncertainty without assuming workers are necessarily risk neutral.

5.2 The structure of the set of JS-stable outcomes

Apart from the natural appeal of stability as a solution concept in matching models it also exhibits a very elegant mathematical structure, as the set of stable outcomes forms a lattice under a natural order. A variety of observations then follows. These these observations may have natural counterparts when considering the larger set of JS-stabile outcomes:

- A basic question we have not tackled refers to the maximal class of production functions which guarantees the existence of a JS-stable outcome (albeit not necessarily an efficient one).
- The set of stable outcomes in many matching models, and in particular in many-to-many matching models, under the gross substitutes assumption, has a natural partial order for which this set is a lattice. Is there a similar structure for the set of JS-stable outcomes? What kinds of production functions allow for such a lattice structure? We suspect that the answers will typically be negative but have not studied this in depth so far.
- A central corollary one can derive from the lattice structure of stable outcomes is the existence of 'best' and 'worst' stable outcomes for the firms as well as for the workers. However, such best and worst allocations may exist even without a lattice structure (e.g., see Hatfield and Kojima (2010)). Thus, the study of extreme allocations that are JS-stable may take place even prior to our full understanding of the existence of a lattice structure for JS-stable outcomes.

5.3 Matching with contracts

The simple many-to-many matching model that we use was extended by Hatfield and Milgrom (2005) to a model of 'matching with contracts'. In such a model a contract between a firm and an agent may specify various aspects related to employment, beyond the salary. It may specify working hours, shifts, insurance, job description, and many more. Therefore there may exist many possible contracts between a worker and a firm. Consequently, the firm's output (and, consequently, its profit) will not depend only on the set of employees it employs but also on the specific contracts signed between the employees and the firm. Milgrom and Hatfield extend the results from the many-to-many matching model to the new matching-with-contracts paradigm under the gross-substitutes assumption. Echenique (2012) showed that under the gross substitutes assumption the two models are in fact equivalent and the seemingly multi-dimensional extension from wages to contracts boils down to a single-dimensional set. However, Hartfield and Kojima (2008; 2010) demonstrate the richness of the matching-with-contracts paradigm by extending some of the results beyond the familiar domain of gross substitutes. In particular they propose new notions of substitutability, called bilateral and unilateral substitutes. Sönmez and Switzer (2013) and Sönmez (2013) provide a realistic example of a market where these weaker assumptions hold and, as a result, offer new and more efficient allocation mechanisms for such markets.

A major component of our analysis is the domain of production function which we analyze. In particular this domain, AFS, goes far beyond gross substitutes. Thus, the matching-with-contracts model is indeed a more general model and is not subject to the Echenique critique. Studying JS-stability in such a model is left to future research.

5.4 Auction theory

As we have noted, our model has a natural connection with combinatorial auctions, where the buyers (the "firms") who have valuations for subsets of goods need to pay the sum of item prices (the "wages") in a bundle in order to keep the bundle. Our solution concept of JS-stability is then a relaxation of the Walrasian equilibrium in the auction context. A Walrasian equilibrium consists of allocations and item prices such that no bidder can improve her utility by adding items or dropping items or doing both. In comparison, equilibria corresponding to JS-stability would not allow bidders to drop items, although they are allowed to add items at current prices. If we call such equilibria *conditional*, all of our results can be translated as properties of conditional equilibria. For example, the social welfare at a conditional equilibrium, whenever it exists, is always a 2-approximation to the optimal; the AFS valuation class is maximal with respect to the property that any welfare-optimal allocation can be supported as a conditional equilibrium. The SPIB game that we discussed in Section 4.4 corresponds to a natural *simultaneous second-price item auction*. In this auction, all bidders simultaneously submit their bids on all items, and then each wins the items on which she bids highest and pays the sum of second highest bids on those items. Previous work (Christodoulou et al., 2008; Bhawalkar and Roughgarden, 2011) showed that allocations at Nash equilibria with no overbidding in this auction give 2-approximation to the optimal social welfare when bidders' valuations are complement free. Our results immediately imply that for *all* valuations, as long as a Nash equilibrium under weak no-overbidding exists, it achieves a 2-approximation to the optimal social welfare. Our study also sheds light on the question of which valuation classes guarantee the existence of pure Nash equilibria in the simultaneous item auction.

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